Int. J. Solids Structures Vol. 27, No. 6, pp. 671–680, 1991
Printed in Great Britain.
0020-7683/91 $3.00 + .00
Pergamon Press plc

A SIMPLE TECHNIQUE FOR FINDING EFFECTIVE ELASTIC CONSTANTS OF CRACKED SOLIDS FOR ARBITRARY CRACK ORIENTATION STATISTICS

C. M. SAYERS
Koninklijke/Shell Exploratie en Produktie Laboratorium, 2288 GD Rijswijk ZH, The Netherlands

and

M. KACHANOV
Department of Mechanical Engineering, Tufts University, Medford, MA 02155, U.S.A.

(*Received* 1 *August* 1989; *in revised form* 12 *January* 1990)

Abstract—The effective elastic properties of cracked solids are anisotropic if the cracks have preferred orientations. In this paper a simple scheme for evaluating the elastic stiffness tensor for an arbitrary orientation distribution of cracks at finite crack densities is presented. The approach is based on a tensorial transformation of the effective elastic constants for isotropic orientation statistics through the use of a second-order crack density tensor.

1. INTRODUCTION

The effective elastic properties of cracked solids are anisotropic if the cracks have preferred orientations. Such a situation is common in rocks, for example, where preferred orientations reflect the stress history of the rock (stress-induced anisotropy). In the approximation of non-interacting cracks the calculation of the effective elastic properties is straightforward for arbitrary orientation statistics: each crack is assumed to be subjected only to the externally applied stress field $\boldsymbol{\sigma}'$ and the contributions to the overall strain from individual cracks are simply summed up. Such results, however, are limited to small crack densities. For higher crack densities when crack interactions cannot be neglected, O'Connell and Budiansky (1974) and Budiansky and O'Connell (1976) proposed a self-consistent scheme for the calculation of the elastic stiffness tensor for random crack orientation statistics. In this method the effect of crack interactions is included by assuming that each crack is embedded in a medium with the effective stiffness of the cracked body. This scheme was extended by Hoenig (1979) to crack distributions for which the overall elastic stiffness tensor is transversely isotropic. Bruner (1976) and Henyey and Pomphrey (1982) have pointed out that the self-consistent scheme may overestimate the crack interactions and have proposed an alternative, differential scheme in which the crack density is increased in small steps and the elastic properties are recalculated incrementally. Finally, Hudson (1980, 1981, 1986) has given results for both randomly orientated and parallel cracks that are correct to second order in the crack density; his results, however, are restricted to moderately small crack densities as shown below. Another method has recently been proposed by Kachanov (1987a) for finding the effective properties for solids having interacting cracks with arbitrary crack interactions that yields accurate analytical results up to high crack densities. However, these results are obtained for each given arrangement of cracks rather than in statistical terms. Thus the only schemes available at present for calculating the effective elastic constants at finite crack densities are the self-consistent and differential schemes.

Both of these schemes are difficult to implement in the case of arbitrary crack orientation statistics, since they are based on placing an individual crack into an anisotropic "effective" matrix and therefore require as input the solution for a single crack in an anisotropic medium, with arbitrary orientation of the crack with respect to the axes of anisotropy. Such solutions are not easily obtainable in analytic form, particularly in the 3-D case. A further problem is the necessity of specifying *a priori* the type and orientation of the effective anisotropy; this may be obvious for the simplest orientation statistics such as parallel cracks but is far less obvious for more complex orientation distributions, a simple example being two sets of non-orthogonal cracks with different densities. Another problem, in the case of the differential scheme, is path dependence: the results may depend on the order in which cracks of different orientations are introduced into the medium.

In this paper a very simple scheme for finding the effective elastic properties of solids for arbitrary orientation statistics at finite crack densities is presented. Besides its computational simplicity the main advantage of this scheme when compared with those mentioned above is that an *a priori* knowledge of the symmetry axes of the elastic tensor of the cracked medium is not required. The approach is based on a tensorial transformation of the effective elastic constants for randomly oriented penny-shaped cracks, which are assumed to be known, to the arbitrary orientation statistics through the use of a second-order crack density tensor $\boldsymbol{\alpha}$ characterizing the averaged geometry of the crack array (Kachanov, 1980, 1987b; Vakulenko and Kachanov, 1971).

2. EXISTING APPROACHES

The elastic constants for an isotropic homogeneous material containing elliptical cracks can be derived from the solution for elliptical cavities as described below. The results will then be specialized to the case of circular cracks for which a description of the elastic constants in terms of the second-order crack density tensor $\boldsymbol{\alpha}$ used in Section 3 is appropriate. As is well known (Hill, 1963), the average strain tensor in the solid for a homogeneous material of volume V containing arbitrary cavities is

$$\frac{1}{V_s}\int_{V_s}\varepsilon_{ij}\,\mathrm{d}V = \bar{\varepsilon}_{ij} + \frac{1}{2V_s}\sum_r\int_{S_r}(u_i n_j + u_j n_i)\,\mathrm{d}S \tag{1}$$

where V_s is the volume of the solid, S_r is the surface of the rth-cavity lying within V and $\bar{\varepsilon}_{ij}$ is the macroscopic strain defined by

$$\bar{\varepsilon}_{ij} = \frac{1}{2V_s}\int_{S_e}(u_i n_j + u_j n_i)\,\mathrm{d}S.$$

Here S_e is the solid portion of the exterior boundary. The macroscopic strains $\bar{\varepsilon}_{ij}$ are related to the macroscopic stress components $\bar{\sigma}_{ij}$ (defined by $\bar{\sigma}_{ij} = (1/V)\int_{V_s}\sigma_{ij}\,\mathrm{d}V$) by the effective compliance tensor M_{ijkl}:

$$\bar{\varepsilon}_{ij} = M_{ijkl}\bar{\sigma}_{kl}.$$

Since ε_{ij} within the solid is given by $\varepsilon_{ij} = M^0_{ijkl}\sigma_{kl}$, where M^0_{ijkl} is the compliance tensor of the solid, eqn (1) gives

$$M_{ijkl}\bar{\sigma}_{kl} = M^0_{ijkl}\bar{\sigma}_{kl}V/V_s + N_{ijkl}\bar{\sigma}_{kl}$$

where

$$N_{ijkl}\bar{\sigma}_{kl} = -\frac{1}{2V_s}\sum_r\int_{S_r}(u_i n_j + u_j n_i)\,\mathrm{d}S.$$

For an ellipsoidal cavity with principal axes $2a \geqslant 2b \geqslant 2c$ it is convenient to introduce a set of axes $Ox'_1x'_2x'_3$ with origin at the centre of the ellipsoid and Ox'_1, Ox'_2 and Ox'_3 along the a, b and c axes, respectively. In the limit $c/a \ll 1$, $c/b \ll 1$ (flat cracks) the above integral may be evaluated by taking the integration over the surface S_r of the crack and replacing displacements by displacement jumps across S_r while putting $V_s = V$. Hence in the crack reference frame the contribution of the rth-crack to $\bar{\varepsilon}_{ij}$ is:

$$(1/2V)\int_{S_r}([u_i']n_j'+[u_j']n_i')\,\mathrm{d}S = N'_{ijkl}\bar{\sigma}'_{kl} \tag{2}$$

where the bracket [ ] denotes jump discontinuities in the displacement. In this limit, $N_{ijkl} = \Delta M_{ijkl}$, the change in M_{ijkl} due to the presence of the cracks.

Under the action of uniform far stresses, an arbitrary ellipsoidal cavity characterized by semi-axes a, b, c in an anisotropic material deforms into another ellipsoid (Eshelby, 1957). Thus the crack face displacements will also be ellipsoidal, so that the cavity displacements u_i are

$$u_i' = \beta_i a(1-x_1'^2/a^2-x_2'^2/b^2)^{1/2}$$

(Hoenig, 1978) where the β_i are dimensionless parameters. For a penny-shaped crack of radius a in an isotropic medium, substitution of this result in eqn (2) gives:

$$\begin{aligned} N'_{3333} &= 16(1-\nu^2)a^3/3E \\ N'_{1313} &= N'_{2323} = 8(1-\nu^2)a^3/(3E(2-\nu)), \end{aligned} \tag{3}$$

all other components of N'_{ijkl} being zero. E and ν are the Young's modulus and Poisson's ratio of the matrix material, and the expressions for the β_i derived by Hoenig (1978) have been used.

In the self-consistent method each crack is assumed to be embedded in a medium with the effective stiffness of the cracked medium. E and ν in eqn (3) are therefore replaced by $\bar{E}$ and $\bar{\nu}$, and the results are averaged over all crack orientations after transforming the N'_{ijkl} back to the unprimed system. For an isotropic random distribution of cracks this gives:

$$\begin{aligned} \bar{E}/E &= 1-16\rho(1-\bar{\nu}^2)(10-3\bar{\nu})/(45(2-\bar{\nu})) \\ \rho &= 45(\nu-\bar{\nu})(2-\bar{\nu})/(16(1-\bar{\nu}^2)(10\nu-\bar{\nu}(1+3\bar{\nu}))) \end{aligned}$$

where $\rho = Na^3/V$ (N is the number of cracks in V) is the crack density. This result was given by O'Connell and Budiansky (1974) and Budiansky and O'Connell (1976).

The differential scheme is obtained by adding the changes N_{ijkl} in an incremental fashion and recalculating the matrix constants M^0_{ijkl} at each increment; this results in a set of two differential equations for E and ν as functions of ρ (Bruner, 1976; Henyey and Pomphrey, 1982).

Figure 1 compares the predictions of the self-consistent and differential schemes for isotropic crack orientation statistics. Figure 1 also shows the results obtained by Hudson's scheme (1981) which is correct to second-order in the crack density. Note that the latter scheme clearly favours the differential scheme at small to moderate crack densities. A notable feature of the self-consistent scheme is the prediction of a vanishing elastic stiffness at a crack density $\rho = na^3 = 9/16$. O'Connell and Budiansky (1974) and Budiansky and O'Connell (1976) argue that the vanishing of the elastic stiffnesses corresponds to a loss of coherence of the solid produced by an intersecting crack network at a critical value of the crack density parameter of 9/16. Bruner (1976) argues that since the self-consistent method treats the material as containing non-intersecting cracks in an elastic continuum, it cannot be expected to predict a loss of coherence. Recent calculations (Charlaix, 1986) have shown that the percolation threshold of a 3-D assembly of widthless discs is given by $\rho = 0.185$. Thus the crack density at which the elastic stiffnesses vanishes in the self-consistent scheme is much higher than the density at which a percolating crack network first forms. This can be expected since the formation of a percolating network does not necessarily mean that the solid loses elastic stiffness.

The extension of the self-consistent scheme to crack distributions for which the effective elastic stiffness is transversely isotropic has been carried out by Hoenig (1979). Figure 2 compares the self-consistent and differential schemes for the case when the crack normals

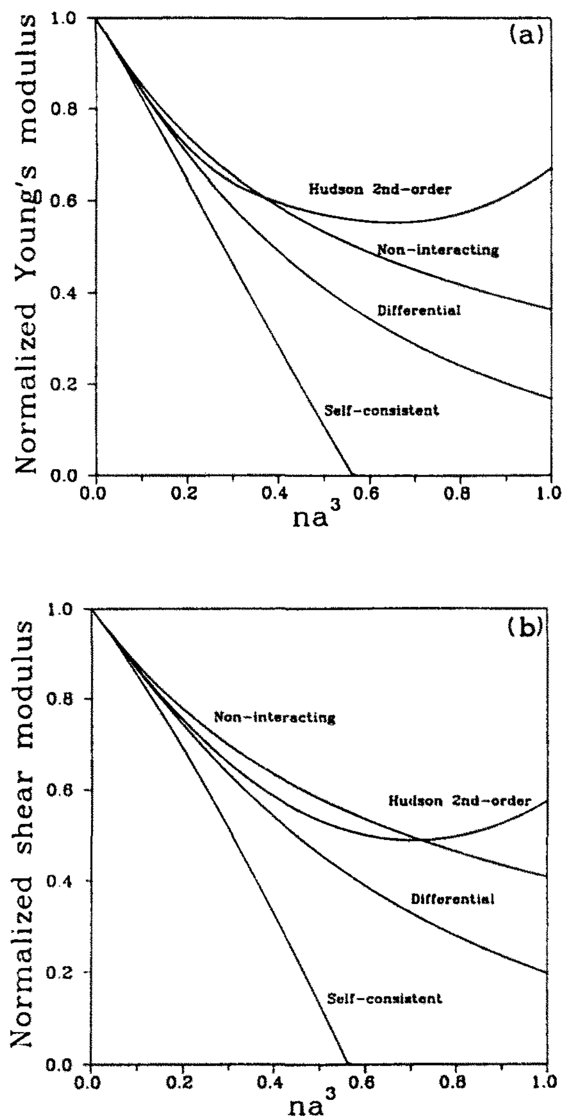

Fig. 1. (a) The normalized Young's modulus $\bar{E}/E$ and (b) the normalized shear modulus $\bar{G}/G$ for an isotropic distribution of penny-shaped cracks in a medium with Poisson's ratio 0.25, computed using the self-consistent, differential and second-order Hudson schemes. Also shown is the result for non-interacting cracks.

are all parallel to Ox_3. In this case it follows from eqn (3) that the only elastic compliances altered by the presence of cracks are $M_{3333} = 1/\bar{E}$, and $M_{3131} = M_{2323} = 1/4\bar{G}$ in the notation of Hoenig (1979). It is seen from Fig. 2 that there is no finite crack density at which $\bar{E}$ or $\bar{G}$ as defined above vanish as would be expected from the vanishing probability of crack intersections for parallel cracks. The Hashin-Shtrikman upper bound for parallel cracks has been given by Laws and Dvorak (1987) and coincides with the non-interacting result shown in Fig. 2. Both the self-consistent and differential schemes give elastic stiffnesses that fall below the upper bound in agreement with the works of Milton (1984) and Norris (1985), who find that both the self-consistent and differential schemes are realizable in the sense that a microstructure can be specified having the elastic constants given by either of the schemes.

The elastic constants for cracks with normals all parallel to a given plane but otherwise randomly distributed have also been evaluated in the self-consistent scheme by Hoenig (1979). Despite the high probability for crack intersections for this orientation statistics, no finite crack density was found for which the effective elastic constants vanish, in contrast with the case of isotropic crack distributions. This inconsistency makes the applicability

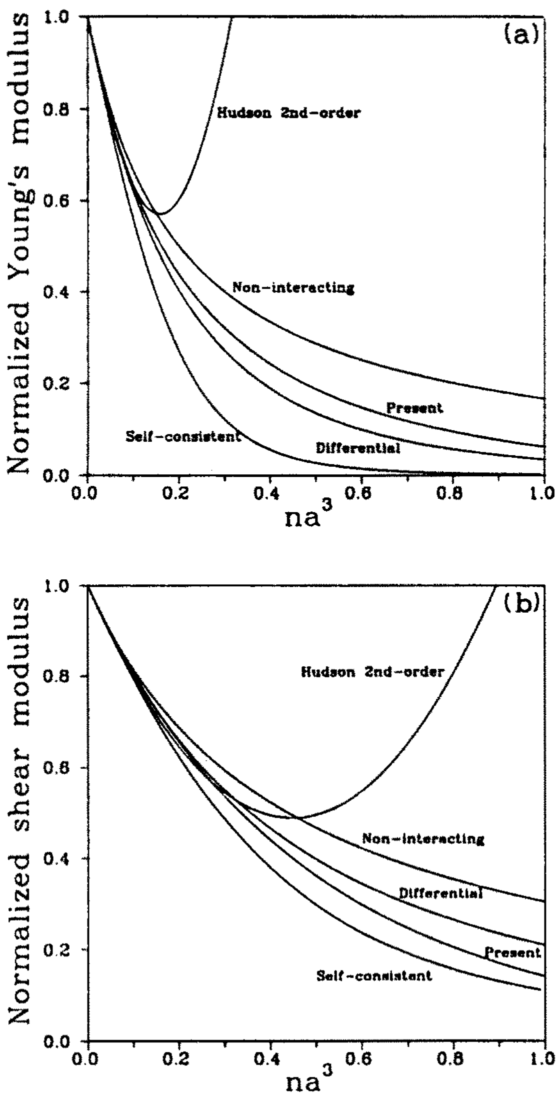

Fig. 2. (a) The normalized Young's modulus $\bar{E}/E$ and (b) the normalized shear modulus $\bar{G}/G$ in the notation of Hoenig (1979) for a distribution of parallel penny-shaped cracks in a medium with Poisson's ratio 0.25, computed using the self-consistent, differential and second-order Hudson schemes. Also shown is the result for non-interacting cracks and the scheme presented in this paper.

of the self-consistent scheme to orientation statistics other than perfectly parallel cracks somewhat questionable. Furthermore, since the results of Hudson (1981), shown in Fig. 1, are correct to second-order in ρ, they support the use of the differential scheme rather than the self-consistent scheme. For these reasons we use the predictions of the differential scheme as input to the method described below.

3. USE OF THE CRACK DENSITY TENSOR

Following Vakulenko and Kachanov (1971) and Kachanov (1980, 1987b) we introduce a symmetric second-order crack density tensor $\boldsymbol{\alpha}$ defined by:

$$\boldsymbol{\alpha} = (1/V)\sum_r \gamma_r \mathbf{n}_r \mathbf{n}_r$$

where V is the volume of averaging, $\mathbf{n}_r$ is the unit normal to the rth-crack and γ_r is a weighting factor characterizing the contribution of the rth-crack to $\boldsymbol{\alpha}$ and depending on the physical problem of interest. In the case of effective elastic properties $\gamma_r = a_r^3$, where a_r

is the radius of the rth-crack for a 3-D solid with penny-shaped cracks (for a 2-D solid with slit-like cracks, $\gamma_r = l_r^2$, where l_r is the semi-length of the rth-crack and V should be changed to the area of averaging). Note that $\alpha_{ii} = \mathrm{tr}\,\boldsymbol{\alpha}$ coincides with the conventional scalar crack density $\rho = n\langle a^3\rangle$ so that $\boldsymbol{\alpha}$ represents a tensorial generalization of ρ accounting for the crack orientation statistics.

The effective elastic compliances M_{ijkl} can be derived from an elastic potential f:

$$\varepsilon_{ij} = \partial f/\partial\sigma_{ij} = M_{ijkl}\sigma_{kl}$$

so that the problem is reduced to finding f. We assume that, in addition to being a function of stress $\boldsymbol{\sigma}$, f is also a function of $\boldsymbol{\alpha}$. If the material is isotropic in the absence of cracks, f will be an isotropic function of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$, i.e. will not change if both $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ undergo the same orthogonal transformation. This implies that $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ will enter f through their invariants only (including the simultaneous ones). Since the stress–strain relations are linear at constant $\boldsymbol{\alpha}$, $f(\boldsymbol{\sigma},\boldsymbol{\alpha})$ must be quadratic in $\boldsymbol{\sigma}$. The resulting expression for f comprises nine terms representing all independent combinations of the invariants (Kachanov 1980, 1987b; Vakulenko and Kachanov, 1971):

$$f(\boldsymbol{\sigma},\boldsymbol{\alpha}) = \omega_1(\mathrm{tr}\,\boldsymbol{\sigma})^2+\omega_2\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})+\eta_1\mathrm{tr}\,\boldsymbol{\sigma}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})+\eta_2\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})$$
$$+\eta_3(\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}))^2+\eta_4(\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}))^2+\eta_5\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})$$
$$+\eta_6\,\mathrm{tr}\,\boldsymbol{\sigma}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})+\eta_7\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}) \quad (4)$$

where a dot indicates one index contraction: $(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})_{ij} = \sigma_{ik}\sigma_{kj}$ and the scalar coefficients ω_i and η_i are, generally, functions of the invariants of $\boldsymbol{\alpha}$. It can be shown, however, that ω_1 and ω_2 are constants not depending on $\boldsymbol{\alpha}$. Indeed, the elastic compliance of a medium with cracks $M_{ijkl}(\boldsymbol{\alpha})$ can be represented as the sum of the matrix compliance M^0_{ijkl} and the contribution from the cracks $\Delta M_{ijkl} = \Delta M_{ijkl}(\boldsymbol{\alpha})$. Therefore $f(\boldsymbol{\sigma},\boldsymbol{\alpha}) = \frac{1}{2}\boldsymbol{\sigma}:\mathbf{M}:\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\sigma}:\mathbf{M}^0:\boldsymbol{\sigma}+\frac{1}{2}\boldsymbol{\sigma}:\Delta\mathbf{M}:\boldsymbol{\sigma}$, where the first term, representing the elastic potential in the absence of cracks, must coincide with the first two terms of eqn (4). ω_1 and ω_2 are therefore constants not depending on $\boldsymbol{\alpha}$ and are given by $\omega_1 = -\nu_0/2E_0$, $\omega_2 = (1+\nu_0)/2E_0$, where ν_0 and E_0 are the Poisson's ratio and Young's modulus in the absence of cracks.

4. TENSORIAL LINEARIZATION OF $f(\boldsymbol{\sigma},\boldsymbol{\alpha})$

As a simple scheme we propose a tensorial linearization of $f(\boldsymbol{\sigma},\boldsymbol{\alpha})$ in $\boldsymbol{\alpha}$, so that

$$f(\boldsymbol{\sigma},\boldsymbol{\alpha}) = \tfrac{1}{2}M_{ijkl}\sigma_{ij}\sigma_{kl} = \tfrac{1}{2}M^0_{ijkl}\sigma_{ij}\sigma_{kl}+\eta_1\,\mathrm{tr}\,\boldsymbol{\sigma}\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})+\eta_2\,\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}) \quad (5)$$

where η_1 and η_2 are, generally, functions of the invariants of $\boldsymbol{\alpha}$. In the following, we assume that η_1 and η_2 are functions of the first invariant $\rho = \mathrm{tr}\,\boldsymbol{\alpha}$ of $\boldsymbol{\alpha}$, the orientational dependence of the effective elastic properties being given by the tensorial structure of the last two terms of eqn (5). For the case of non-interacting cracks η_1 and η_2 are constants and the linearized eqn (5) is the exact representation of the elastic potential for 2-D crack arrays and a good approximation for 3-D arrays (Kachanov, 1980). In the framework of the model (4) the effective elastic properties are always orthotropic, with the axes of orthotropy coinciding with the principal axes of $\boldsymbol{\alpha}$; an additional simplification obtained in reducing eqn (4) to (5) is that the orthotropy is of a special type: (a) it is characterized by six (rather than nine) independent constants and (b) the variation of the elastic moduli with orientation is described by an elliptical surface rather than one of fourth order. Note that analysis (Sayers, 1988a,b) of ultrasonic wave velocity measurements in rocks with an anisotropic crack distribution (Thill *et al.*, 1969; Nur and Simmons, 1969) shows the fourth-order variation of the elastic modulus to be an order of magnitude smaller than that of second-order even for rather high crack densities. These observations support the simple structure of eqn (5) for the case of interacting cracks.

In a coordinate system with axes coincident with the principal axes of α the non-vanishing elastic compliances are M_{pqpq} and M_{ppqq}, which are given by

$$M_{pqpq} = M^0_{pqpq} + (2\eta_1 + \eta_2)\alpha_{pq}\delta_{pq} + (\eta_2/2)(\alpha_{pp} + \alpha_{qq})$$
$$M_{ppqq} = M^0_{ppqq} + \eta_1(\alpha_{pp} + \alpha_{qq}) + 2\eta_2\alpha_{pq}\delta_{pq} \tag{6}$$

where the indices p and q are not to be summed over.

$\eta_1 = \eta_1(\rho)$ and $\eta_2 = \eta_2(\rho)$ may be specified by using the elastic compliances obtained with the differential scheme for an isotropic distribution of cracks. The model then yields, in a very simple way, the effective elastic constants for a solid with arbitrary orientation statistics; these results can be compared with the available results for non-random crack orientations obtained by other methods.

4.1. *Randomly orientated cracks*

For an isotropic (random) crack array $\alpha = (\mathrm{tr}\,\alpha/3)\mathbf{I} = \rho\mathbf{I}/3$, where $\rho = na^3$ is the conventional scalar crack density and $\mathbf{I}$ is the unit tensor; then:

$$\Delta M_{1111} = \Delta M_{2222} = \Delta M_{3333} = 2(\eta_1 + \eta_2)\rho/3$$
$$\Delta M_{1212} = \Delta M_{2323} = \Delta M_{3131} = \eta_2\rho/3$$
$$\Delta M_{1122} = \Delta M_{1133} = \Delta M_{2233} = 2\eta_1\rho/3.$$

Hence, if the dependence of the two effective elastic constants M_{1111} and M_{1212} on ρ is known for the case of isotropic crack orientation statistics, the coefficients η_1 and η_2 can be specified as functions of ρ.

The functions $\eta_1(\rho)$ and $\eta_2(\rho)$ are thus determined by the choice of the input for the case of isotropic crack orientation statistics. Hence, the choice of model for this case will affect the predictions of our model for other orientation statistics. For reasons discussed in Section 2 we use the results for the differential scheme as input.

4.2. *Perfectly aligned cracks*

For perfectly aligned cracks with normals along Ox_3, $\alpha_{11} = \alpha_{22} = 0$, $\alpha_{33} = \rho$ and therefore:

$$\Delta M_{1111} = \Delta M_{2222} = \Delta M_{1212} = \Delta M_{1122} = 0$$
$$\Delta M_{3333} = 2(\eta_1 + \eta_2)\rho$$
$$\Delta M_{2323} = \Delta M_{3131} = \eta_2\rho/2$$
$$\Delta M_{1133} = \Delta M_{2233} = \eta_1\rho.$$

Figure 2 compares the predictions of the scheme presented above for perfectly aligned cracks with the moduli calculated directly by the differential scheme. A comparison of the results with those of the Hudson (1980) scheme (which are correct to second order in the crack density) also shown in Fig. 2 shows the scheme to give a substantial improvement over the non-interacting scheme.

4.3. *Cracks with normals lying randomly in parallel planes*

Figure 3 shows the predictions of the scheme for the case of a crack distribution for which the crack normals lie in planes parallel to the x_1x_2 plane. In this case the only elastic constants altered by the presence of cracks are $M_{1111} = M_{2222} = 1/\bar{E}$, $M_{3131} = M_{2323} = 1/4\bar{G}$ and $M_{1212} = 1/4G^*$ in the notation of Hoenig (1979). This case was previously treated by Hoenig (1979) in the self-consistent scheme and could only be evaluated numerically. By contrast, the present scheme gives a very simple result:

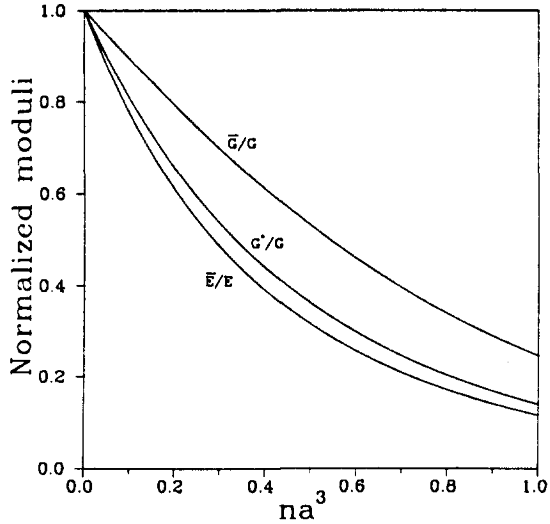

Fig. 3. The normalized Young's modulus $\bar{E}/E$, the normalized shear modulus $\bar{G}/G$ and the normalized shear modulus G^*/G in the notation of Hoenig (1979) for a distribution of penny-shaped cracks with normals randomly distributed in planes parallel to the x_1x_2 plane in a medium with Poisson's ratio 0.25, computed using the scheme presented in this paper.

$$\Delta M_{1111} = \Delta M_{2222} = (\eta_1+\eta_2)\rho$$
$$\Delta M_{3333} = 0$$
$$\Delta M_{1212} = \eta_2\rho/2$$
$$\Delta M_{2323} = \Delta M_{3131} = \eta_2\rho/4$$
$$\Delta M_{1122} = \eta_1\rho$$
$$\Delta M_{1133} = \Delta M_{2233} = \eta_1\rho/2.$$

In agreement with Hoenig (1979), $\bar{G}/G > G^*/G > \bar{E}/E$ (Fig. 3).

4.4. *Two sets of non-orthogonal cracks with different densities*

The use of the present scheme for other crack orientation statistics is equally simple. As an example, consider two sets of non-orthogonal cracks with different densities. The treatment of such a case by the differential scheme is difficult since it requires the knowledge of the solution for one crack in an anisotropic matrix with an acute angle between the crack normal and the axes of anisotropy. The self-consistent scheme requires, in addition, an *a priori* knowledge of the orientation of the axes of anisotropy. Denoting by $\mathbf{n}_a$ and $\mathbf{n}_b$ the normals to set a and b with densities ρ_a and ρ_b, the crack density tensor $\boldsymbol{\alpha} = \rho_a\mathbf{n}_a\mathbf{n}_a+\rho_b\mathbf{n}_b\mathbf{n}_b$. It is convenient to choose a coordinate system with x_3 along $\mathbf{n}_a\times\mathbf{n}_b$ and x_1 and x_2 in the $\mathbf{n}_a\mathbf{n}_b$ plane. Choosing x_1 and x_2 as the principal directions of $\boldsymbol{\alpha}$, we find $\alpha_{11} = A/2$, $\alpha_{22} = B/2$, where

$$A = (\rho_a+\rho_b)+\{(\rho_a+\rho_b)^2-4\rho_a\rho_b\sin^2\phi\}^{1/2}$$
$$B = (\rho_a+\rho_b)-\{(\rho_a+\rho_b)^2-4\rho_a\rho_b\sin^2\phi\}^{1/2}$$

and ϕ is the angle between $\mathbf{n}_a$ and $\mathbf{n}_b$. This gives

$$\Delta M_{1111} = (\eta_1+\eta_2)A$$
$$\Delta M_{2222} = (\eta_1+\eta_2)B$$
$$\Delta M_{3333} = 0$$
$$\Delta M_{1212} = \eta_2\rho/2$$

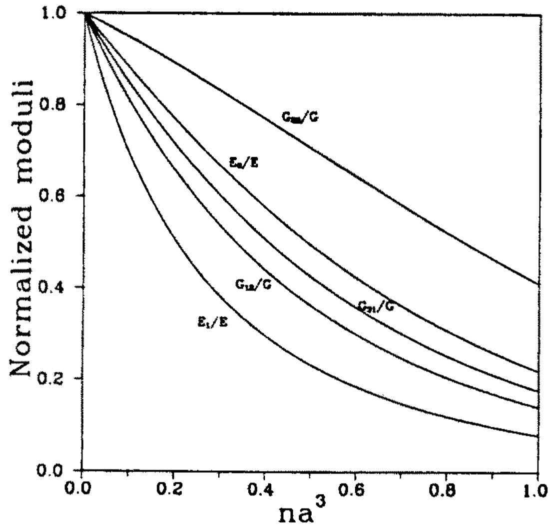

Fig. 4. The normalized elastic stiffnesses for two sets of non-orthogonal cracks with densities $\rho_a = 3\rho/5$, $\rho_b = 2\rho/5$ and $\phi = 60°$ in a medium with Poisson's ratio 0.25 computed using the scheme presented in this report. The subscripts refer to the principal axes of the crack density tensor $\boldsymbol{\alpha}$.

$$\Delta M_{1313} = \eta_2 A/4$$
$$\Delta M_{2323} = \eta_2 B/4$$
$$\Delta M_{1122} = \eta_1 \rho$$
$$\Delta M_{1133} = \eta_1 A/2$$
$$\Delta M_{2233} = \eta_1 B/2.$$

As a numerical illustration, consider the case where $\phi = 60°$, $\rho_a = 3\rho/5$, $\rho_b = 2\rho/5$. The resultant elastic constants, in the notation $E_1 = 1/M_{1111}$, $E_2 = 1/M_{2222}$, $G_{12} = 1/4M_{1212}$, $G_{31} = 1/4M_{3131}$, $G_{23} = 1/4M_{2323}$, are shown in Fig. 4.

5. DISCUSSION AND CONCLUSIONS

A simple scheme for evaluating the elastic stiffness tensor for arbitrary orientation statistics at finite crack densities has been presented. The scheme is based on a tensorial transformation of the effective elastic constants for randomly orientated cracks through the use of a second-order crack density tensor $\boldsymbol{\alpha}$ characterizing the averaged geometry of the crack array. The comparison of the scheme with the second-order approach of Hudson shows the scheme to be a considerable improvement on the non-interacting approach whilst it maintains the computational simplicity of that approach. In particular, an *a priori* knowledge of the symmetry axes of the elastic tensor of the cracked medium is not required.

The scheme proposed allows generalizations and refinements. For example, more than two of the coefficients $\eta_1, \ldots, \eta_7$ may be retained and determined, as functions of ρ, by using known results for random, parallel and other crack orientation statistics. These results may not be very reliable at high crack densities, however, so that their use may not necessarily improve the accuracy of the scheme. In addition, such refinements will result in extra terms in the elastic potential (5) thus making it less simple.

Acknowledgements—This paper was published with the permission of Shell Research B.V. The authors thank F. K. Lehner (K.S.E.P.L.) for helpful discussions.

REFERENCES

Bruner, W. M. (1976). Comment on "Seismic velocities in dry and saturated cracked solids" by R. J. O'Connell and B. Budiansky. *J. Geophys. Res.* **81**, 2573–2576.

Budiansky, B. and O'Connell, R. J. (1976). Elastic moduli of a cracked solid. *Int. J. Solids Structures* **12**, 81–97.
Charlaix, E. (1986). Percolation threshold of a random array of discs: a numerical simulation. *J. Phys. A* **19**, L351–L354.
Eshelby, J. D. (1957). The determination of the elastic field of an ellipsoidal inclusion and related problems. *Proc. R. Soc.* **A241**, 376–396.
Henyey, F. S. and Pomphrey, N. (1982). Self-consistent moduli of a cracked solid. *Geophys. Res. Lett.* **9**, 903–906.
Hill, R. (1963). Elastic properties of reinforced solids: some theoretical principles. *J. Mech. Phys. Solids* **11**, 357–372.
Hoenig, A. (1978). The behaviour of a flat elliptical crack in an anisotropic body. *Int. J. Solids Structures* **14**, 925–934.
Hoenig, A. (1979). Elastic moduli of a non-randomly cracked body. *Int. J. Solids Structures* **15**, 137–154.
Hudson, J. A. (1980). Overall properties of a cracked solid. *Math. Proc. Camb. Phil. Soc.* **88**, 371–384.
Hudson, J. A. (1981). Wave speeds and attenuation of elastic waves in material containing cracks. *Geophys. J. R. Astr. Soc.* **64**, 133–150.
Hudson, J. A. (1986). A higher order approximation to the wave propagation constants for a cracked solid. *Geophys. J. R. Astr. Soc.* **87**, 265–274.
Kachanov, M. (1980). Continuum model of medium with cracks. *J. Engng Mech. Div. ASCE* **106**, 1039–1051.
Kachanov, M. (1987a). Elastic solids with many cracks: a simple method of analysis. *Int. J. Solids Structures* **23**, 23–43.
Kachanov, M. (1987b). On modelling of anisotropic damage in elastic-brittle materials—a brief review. In *Damage Mechanics in Composites* (Edited by A. Wang and G. Haritos), pp. 99–105. ASME Winter Annual Meeting, Boston 1987.
Laws, N. and Dvorak, G. J. (1987). The effect of fibre breaks and aligned penny-shaped cracks on the stiffness and energy release rates in unidirectional composites. *Int. J. Solids Structures* **23**, 1269–1283.
Milton, G. (1984). The coherent potential approximation is a realizable effective medium scheme. Preprint.
Norris, A. N. (1985). A differential scheme for the effective moduli of composites. *Mech. Mater.* **4**, 1–16.
Nur, A. and Simmons, G. (1969). Stress-induced velocity anisotropy in rock: an experimental study. *J. Geophys. Res.* **74**, 6667–6674.
O'Connell, R. J. and Budiansky, B. (1974). Seismic velocities in dry and saturated cracked solids. *J. Geophys. Res.* **79**, 5412–5426.
Sayers, C. M. (1988a). Inversion of ultrasonic wave velocity measurements to obtain the microcrack orientation distribution function in rocks. *Ultrasonics* **26**, 73–77.
Sayers, C. M. (1988b). Stress-induced ultrasonic wave velocity anisotropy in fractured rock. *Ultrasonics* **26**, 311–317.
Thill, R. E., Willard, R. J. and Bur, T. R. (1969). Correlation of longitudinal velocity variation with rock fabric. *J. Geophys. Res.* **74**, 4897–4909.
Vakulenko, A. A. and Kachanov, M. (1971). Continuum theory of medium with cracks. *Mehanika Tverdogo Tela* **6**, 159–166 (English translation: *Mech. Solids* **6**, 145–151 (1971)).